The Rectilinear Crossing Number of a Complete Graph and Sylvester's "Four Point Problem" of Geometric Probability

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The chance of ... the quadrilateral formed by joining four points, taken arbitrarily within any assigned boundary, constituting a reentrant or convex quadrilateral, will serve as types of the class of questions in view.

—J. J. Sylvester [**11**]

We prove that two fundamental constants of the geometry of the plane are equal.

First, if R is an open set in the plane with finite Lesbesque measure, let $q(R)$ denote the probability that if four points are chosen independently uniformly at random in R, then their convex hull is a quadrilateral. Let q_* be the infimum of $q(R)$ over all such sets R.

Second, let $\bar{\nu}(K_n)$ denote the rectilinear crossing number of the complete graph on n vertices, i.e., the minimum number of intersections in any drawing of K_n in the plane that has straight-line-segment edges. It is well known that $\bar{\nu}(K_n)/\binom{n}{4}$ increases steadily to some limit $\bar{\nu}^*$ as $n \to \infty$.

Our main result is that $q_* = \bar{\nu}^*$.

FOUR RANDOM POINTS. Let R be an open set in the plane with finite area. As such, we can consider R to be a sample space from which we select points independently uniformly at random (i.u.a.r.). Choose four points from R i.u.a.r.. Then with probability 1, no three of the points are collinear, so the convex hull of the four points is either a triangle (one point in the convex hull of the other three) or a quadrilateral. J. J. Sylvester [**11**] asked, what is the probability that the points determine a convex quadrilateral? We denote this probability by $q(R)$.

How large and how small can $q(R)$ be? When R is restricted to being a convex set we have the following result (see Blaschke [**1, 2**] and also [**7**]).

Theorem 1. *Let R be an open,* convex *subset of the plane, of finite area. Then*

$$\frac{2}{3} \leq q(R) \leq 1 - \frac{35}{12\pi^2} \approx 0.704.$$

Further, both inequalities are sharp. The lower bound is attained by the interior of a triangle (any triangle), and the upper bound by the interior of an ellipse (any ellipse). □

This theorem, however, does not fully address the issue of the extreme values of $q(R)$ because it considers only *convex* regions R. It is easy to see that if we relax the convexity requirement, then the supremum of $q(R)$ is 1; let R be a very thin open annulus and observe that we can make $q(R)$ arbitrarily close to 1. Thus it remains to consider the infimum of $q(R)$; let $q_* = \inf q(R)$ where the infimum is over all open sets R with finite area.

We show below that q_* is positive and strictly less than $2/3$ (the lowest possible result for convex R). We show further that q_* is closely related to the rectilinear crossing number of complete graphs.

RECTILINEAR CROSSING NUMBER OF A GRAPH. Let G be a graph which we wish to draw in the plane. If G is planar, then we can find an embedding in which the edges do not cross. A result of Fáry [**4**] shows that we can choose this embedding so that the edges are noncrossing straight line segments.

Let $\bar{\nu}(G)$ denote the minimum number of crossings in a straight line drawing of G in the plane; the parameter $\bar{\nu}(G)$ is known as the *rectilinear crossing number* of G. (For background on the rectilinear crossing number, see [**6**], [**7**] or [**12**].)

An important open problem in the study of graph embeddings is to determine the rectilinear crossing number of the complete graph K_n. For $n = 5, 6, 7, 8, 9$ the values are known (see [**12**]) and they are $1, 3, 9, 19, 36$ respectively. For $n = 10$ it is known [**10**] that $61 \leq \bar{\nu}(K_{10}) \leq 62$.

If we place the n vertices of K_n on a circle, then the number of crossings is exactly $\binom{n}{4}$; certainly we can do better, but $\bar{\nu}(K_n)$ is on the order of n^4 as we now explain.

Theorem 2. *There exists a constant $\bar{\nu}^*$ such that $0 < \bar{\nu}^* < \infty$ and*

$$\bar{\nu}^* = \lim_{n \to \infty} \frac{\bar{\nu}(K_n)}{\binom{n}{4}} = \sup_n \frac{\bar{\nu}(K_n)}{\binom{n}{4}}.$$

(This is well-known folklore, but for completeness we show the proof here.)

Proof: Let $m < n$ and consider a straight line embedding of K_n in the plane with the minimum number of crossings, $\bar{\nu}(K_n)$. For each m element subset A of $V(K_n)$, let $c(A)$ denote the number of crossings in this embedding in which the endpoints of the crossing edges are all in A. If we sum $c(A)$ over all m-subsets of $V(K)$, we count each possible crossing exactly $\binom{n-4}{m-4}$ times. Thus

$$\bar{\nu}(K_n) = \sum_{|A| = m} c(A) \Big/ \binom{n-4}{m-4}.$$

Now clearly $c(A) \geq \bar{\nu}(K_m)$, so it follows that

$$\bar{\nu}(K_n) \geq \frac{\binom{n}{m}}{\binom{n-4}{m-4}} \bar{\nu}(K_m)$$

which we can rearrange to read

$$\frac{\bar{\nu}(K_n)}{\binom{n}{4}} \geq \frac{\bar{\nu}(K_m)}{\binom{m}{4}}.$$

Thus $\bar{\nu}(K_n)/\binom{n}{4}$ is a nondecreasing function of n which is bounded above by 1 and below by $\bar{\nu}(K_5)/\binom{5}{4} = 1/5$. $\square$

Since $\bar{\nu}(K_{10}) \geq 61$, we see that $\bar{\nu}^* \geq 61/210 \approx 0.29$.

Singer [10] proves the following upper bound on the rectilinear crossing number of K_n when n is a power of 3:

$$\bar{\nu}(K_n) \leq \frac{1}{312}(5n^4 - 39n^3 + 91n^2 - 57n).$$

Thus $\bar{\nu}^* \leq \frac{5}{312} \times 24 = \frac{5}{13} \approx 0.3846$.

(Jensen [6] gives a rectilinear embedding of K_n with $\frac{7}{432}n^4 + O(n^3)$ crossings, yielding an upper bound of 0.3888... on $\bar{\nu}^*$.)

MAIN RESULT. Our main result is a simple relation between q_*, the smallest probability of choosing a quadrilateral, and $\bar{\nu}^*$, the limit of $\bar{\nu}(K_n)/\binom{n}{4}$.

Theorem 3. *With the preceding notation*, $q_* = \bar{\nu}^*$.

Proof: Let R be any open set in the plane with finite area. Choose n points $p_1, p_2, \ldots, p_n$ i.u.a.r. from R and let those n points be the vertices of a straight line drawing of K_n. Let c be the number of crossings in this drawing. Now c is a random variable whose value is always at least $\bar{\nu}(K_n)$. Further, let

$$X = \sum_{\{a,b,c,d\}} \mathbf{1}\{p_a, p_b, p_c, p_d \text{ form a quadrilateral}\}$$

where the sum is over all 4 element subsets of $\{1, \ldots, n\}$ and $\mathbf{1}\{\ldots\}$ is a 0, 1 indicator random variable whose value is 1 just when the convex hull of the four points p_a, p_b, p_c, p_d is a quadrilateral. Since the optimum drawing cannot have more crossings than the average, we get, by taking expectations,

$$\bar{\nu}(K_n) \leq E(X) = \binom{n}{4} q(R)$$

for all n. Dividing by $\binom{n}{4}$ and letting $n \to \infty$, we have $\bar{\nu}^* \leq q(R)$ for all R. Thus $\bar{\nu}^* \leq q_*$.

For the opposite inequality, consider a straight line embedding of K_n with the minimum number, $\bar{\nu}(K_n)$, of crossings. Let R_ϵ be the (disconnected) open set formed by placing a small open disc of radius ϵ centered at each vertex of the embedding. See the cover of this issue. Here ϵ is chosen small enough so that for every choice of n points, one in each disc, if we connect all pairs of them by straight line segments then the number of crossings is always equal to $\bar{\nu}(K_n)$, i.e., all such embeddings are optimal. Clearly such an ϵ exists.

We now consider the following question: choose four distinct discs of R_ϵ, and then choose i.u.a.r. a point from each of them. What is the probability q that the resulting quadrilateral is convex?

On the one hand, q is the number of convex quadrilaterals in the original embedding divided by $\binom{n}{4}$. But the former are in 1-1 correspondence with edge crossings, so there are exactly $\bar{\nu}(K_n)$ of them, and we have $q = \bar{\nu}(K_n)/\binom{n}{4}$.

On the other hand, $q(R_\epsilon)$ is the probability that four points chosen i.u.a.r. in R_ϵ will form a convex quadrilateral. But four points so chosen will lie in four distinct discs of R_ϵ with probability $1 - O(1/n)$. Hence

$$q = q(R_\epsilon) + O(1/n) \geq q_* + O(1/n).$$

Combining these two facts about q we obtain

$$q = \bar{\nu}(K_n)/\binom{n}{4} \geq q_* + O(1/n).$$

If we now let $n \to \infty$, $\bar{\nu}^* \geq q_*$ follows, and the proof is complete. □

Thus to summarize our principal results, we have

$$0.29 \approx \frac{61}{210} \leq q_* = \bar{\nu}^* \leq \frac{5}{13} \approx 0.385.$$

SOME COMMENTS

1. In a spirit similar to ours, Moon **[8]** applies random methods in bounding the ordinary crossing number of K_n. He places n points i.u.a.r. on a sphere and joins them pairwise by arcs of great circles. This gives an upper bound of $\frac{3}{8}\binom{n}{4}$ for the crossing number of K_n. It is not clear how to project Moon's embedding into the plane and have the edges become line segments.
2. Our methods can be applied to arbitrary graphs G. Let M denote the number of pairs of edges in G which span four distinct vertices. Then $\bar{\nu}(G) \leq \bar{\nu}^* M/3$; simply place the vertices of G i.u.a.r. in a region R and compute the expected number of crossings.
3. Given a subset R of the plane together with a probability measure μ defined on R, define $q(R, \mu)$ to be the probability that four points chosen i.u.a.r. with respect to μ form a convex quadrilateral. Without further restrictions, we see that the infimum of $q(R, \mu)$ is 0; let R be a unit line segment together with Lebesgue measure—no four points can form a quadrilateral.

   If we restrict ourselves to those (R, μ) for which the probability that three points selected i.u.a.r. are collinear is 0, then the infimum of $q(R, \mu)$ remains the same, namely q_*.

   If we understand Sylvester's problem (see quote above) to mean the interior of a Jordan curve, our results still don't change; in the proof of Theorem 3 we can join the small open disks by even smaller tendrils so the domain is the interior of a simple closed curve.
4. Let R be a bounded open set and embed the vertices of K_n at n points selected i.u.a.r. in R. We have seen that the expected number of crossings in this embedding is $q(R)\binom{n}{4}$. However, one might harbor hopes of doing better on occasion. It would seem natural to generate many embeddings of K_n in, say, the interior of a square or a disk, and count the number of crossings in hopes of finding a good embedding. Regrettably, this is not at all likely, as we now explain.

   The number of crossings can be written

$$X = \sum_{\{a,b,c,d\}} \mathbf{1}\{p_a, p_b, p_c, p_d \text{ form a quadrilateral}\}$$

   where the sum is over all 4-element subsets of $\{1, \ldots, n\}$ and the p_i's are chosen i.u.a.r. in R. Thus X is an example of a *U-statistic*; see **[5]** or **[9]** for an extensive discussion. Using "deviations" results (see **[9]** §5.6) one can show that the probability that the number of crossings is "significantly" less than the expectation is extremely small.

REFERENCES

1. W. Blaschke, Über affine Geometrie XI: Lösung des "Vierpunktproblems" von Sylvester aus der Theorie der geometrischen Wahrscheinlichkeiten, *Leipziger Berichte* 69 (1917) 436–453.
2. W. Blaschke, *Vorlesungen über Differentialgeometrie II: Affine Differentialgeometrie*, Springer, Berlin (1923).
3. P. Erdős, and R. K. Guy, Crossing number problems, This *Monthly* 80 (1973) 52–57.
4. I. Fáry, On straight line representations of planar graphs, *Acta Sci. Math. (Szeged)* 11 (1948) 229–233.
5. W. Hoeffding, The strong law of large numbers for *U*-statistics, Univ. North Carolina Institute of Statistics Mimeo Series, no. 302 (1961).
6. H. F. Jensen, An upper bound for the rectilinear crossing number of the complete graph, *J. Comb. Theory (B)* 10 (1971) 212–216.
7. Victor Klee, What is the expected volume of a simplex whose vertices are chosen at random from a given convex body?, This *Monthly* 76 (1969) 286–288.
8. J. W. Moon, On the distribution of crossings in random complete graphs, *SIAM J.* 13 (1965) 506–510.
9. Robert J. Serfling, *Approximation Theorems of Mathematical Statistics*, Wiley (1980).
10. David Singer, The rectilinear crossing number of certain graphs, manuscript (1971).
11. J. J. Sylvester, On a special class of questions on the theory of probabilities, *Birmingham British Association Report* (1865) 8.
12. Arthur T. White, and Lowell W. Beineke, Topological graph theory, in *Selected Topics in Graph Theory*, Beineke & Wilson, eds., Academic (1978) 15–49.

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Biographical history, as taught in our public schools, is still largely a history of boneheads: ridiculous kings and queens, paranoid political leaders, compulsive voyagers, ignorant generals—the flotsam and jetsam of historical currents. The men who radically altered history, the great scientists and mathematicians, are seldom mentioned, if at all.

—*Martin Gardner*
George F. Simmons, *Calculus Gems.*
New York: McGraw Hill, Inc., 1992, p. 1